Multiplicity and stability of closed geodesics on bumpy Finsler 3-spheres

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Abstract

We prove that for every $\mathbf{Q}$-homological Finsler 3-sphere (M, F) with a bumpy and irreversible metric F, either there exist two non-hyperbolic prime closed geodesics, or there exist at least three prime closed geodesics.

1 Introduction and the main result

Let M be a smooth manifold with a Finsler metric F. A continuous curve $c : [0,1] \to M$ on a Finsler manifold (M, F) is a closed geodesic, if c is closed and is the shortest curve connecting any two points on the image of c which are close enough (cf. [BCS1] and [She1]). As usual on any Finsler manifold $M = (M, F)$, a closed geodesic $c : S^1 = \mathbf{R}/\mathbf{Z} \to M$ is *prime*, if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the m-th iteration c^m of c is defined by $c^m(t) = c(mt)$ for $m \in \mathbf{N}$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1-t)$ for $t \in \mathbf{R}$. We call two prime closed geodesics c and d *distinct* if there is no $\theta \in (0,1)$ such that $c(t) = d(t+\theta)$. We shall omit the word "distinct" for short when we talk about more than one prime closed geodesics. A closed geodesic c is *non-degenerate* if 1 is not an eigenvalue of the linearized Poincaré map P_c of c.

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c is *hyperbolic* if $\sigma(P_c) \cap \mathbf{U} = \emptyset$, or *elliptic* if $\sigma(P_c) \subset \mathbf{U}$, where we denote by $\mathbf{U} = \{z \in \mathbf{C} \,|\, |z| = 1\}$. A Finsler metric F is *bumpy* if all closed geodesics and their iterations are all non-degenerate.

Note that by the classical theorem of Lyusternik-Fet [LyF1], there exists at least one closed geodesic on every compact Riemannian as well as Finsler manifold. In [Kat1] of 1973, Katok constructed his famous irreversible Finsler metrics on S^{2n} respectively S^{2n-1} with precisely $2n$ prime closed geodesics all of which are elliptic (cf. also [Zil1]). Based on this result in [Ano1] of 1974, Anosov conjectured that the lower bound of the number of prime closed geodesics on S^n is $2[\frac{n}{2}]$.

We are only aware of a few results on the multiplicity and stability of closed geodesics on Finsler spheres. In 1965, Fet in [Fet1] proved that there exist at least two distinct closed geodesics on every reversible bumpy Finsler manifold (M, F). In 1989, Rademacher in [Rad1] proved that there exist at least two elliptic closed geodesics on every bumpy Finsler 2-sphere with finitely many prime closed geodesics. In 2003, Hofer, Wysocki and Zehnder in [HWZ1] proved that there exist either two or infinitely many prime closed geodesics on every bumpy Finsler 2-sphere if the stable and unstable manifolds of every hyperbolic closed geodesic intersect transversally. In [Rad3] of 2005, Rademacher obtained existences and stability of closed geodesics on Finsler S^n under pinching conditions which generalizes results in [BTZ1] and [BTZ2] of Ballmann, Thorbergsson and Ziller in 1982-83 on Riemannian manifolds. Recently, Bangert and Long in [BaL1] proved that there exist always at least two prime closed geodesics on every Finsler 2-sphere (S^2, F) which answers Anosov's conjecture for S^2. More recently Duan and Long in [DuL1] and Rademacher in [Rad4] proved independently that there exist at least two distinct prime closed geodesics on every bumpy Finsler n-sphere (S^n, F) with $n \geq 3$.

Note that in [Hin1] of 1984, Hingston's result specially implies that on every Riemannian sphere S^n, if all the closed geodesics are hyperbolic, then there exist infinitely many geometrically distinct closed geodesics. In [Rad1] of 1989, Rademacher proved that on every even dimensional bumpy Finsler sphere S^{2n}, if there are only finitely many prime closed geodesics, then at least one of them is non-hyperbolic. In the recent [LoW1], Long and Wang proved that if there exist precisely two prime closed geodesics on a Finsler S^2, both of them must be irrationally elliptic.

Note that Long in [Lon3] conjectured that the number of prime closed geodesics for (S^3, F) may belong to $\{2, 3, 4\} \bigcup \{+\infty\}$. This paper is devoted to the proof of the following main result which is related to this conjecture.

Theorem 1.1. *For every $\mathbf{Q}$-homological Finsler 3-sphere (M, F) with a bumpy and irreversible metric F, either there exist precisely two non-hyperbolic prime closed geodesics, or there exist at*

least three distinct prime closed geodesics.

Theorem 1.1 is a consequence of the slightly stronger version of Theorem 3.6 below. Our proof of these theorems relies mainly on the following ingredients: The precise index iteration formulae of Long, the common index jump theorem of Long and Zhu, the Morse inequalities, and Rademacher's mean index identity for closed geodesics, together with some new techniques relating local and global information. Because the proof for $\mathbf{Q}$-homological 3-spheres is the same as that for S^3, we carry out below the proof only for (S^3, F) with a bumpy and irreversible Finsler metric F. The main idea is that assuming the existence of precisely two prime closed geodesics on (S^3, F) and at least one of them being hyperbolic, we shall derive a contradiction via the above mentioned tools.

In this paper, let $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote the sets of positive integers, non-negative integers, rational numbers, real numbers, and complex numbers respectively. We denote by $[a] = \max\{k \in \mathbf{Z} \,|\, k \le a\}$ for any $a \in \mathbf{R}$. When S^1 acts on a topological space X, we denote by $\overline{X}$ the quotient space X/S^1. We use only singular homology modules with $\mathbf{Q}$-coefficients. For terminologies in algebraic topology we refer to [GrH1].

2 Preliminary results on closed geodesics

2.1 Critical modules of iterations of closed geodesics

Let $M = (M, F)$ be a compact Finsler manifold (M, F), the space $\Lambda = \Lambda M$ of H^1-maps $\gamma : S^1 \to M$ has a natural structure of Riemannian Hilbert manifolds on which the group $S^1 = \mathbf{R}/\mathbf{Z}$ acts continuously by isometries, cf. [Kli2], Chapters 1 and 2. This action is defined by $(s \cdot \gamma)(t) = \gamma(t+s)$ for all $\gamma \in \Lambda$ and $s, t \in S^1$. For any $\gamma \in \Lambda$, the energy functional is defined by

$$E(\gamma) = \frac{1}{2} \int_{S^1} F(\gamma(t), \dot{\gamma}(t))^2 dt. \tag{2.1}$$

It is $C^{1,1}$ (cf. [Mer1]) and invariant under the S^1-action. The critical points of E of positive energies are precisely the closed geodesics $\gamma : S^1 \to M$. The index form of the functional E is well defined along any closed geodesic c on M, which we denote by $E''(c)$ (cf. [She1]). As usual, we denote by $i(c)$ and $\nu(c)$ the Morse index and nullity of E at c. In the following, we denote by

$$\Lambda^\kappa = \{d \in \Lambda \mid E(d) \le \kappa\}, \quad \Lambda^{\kappa-} = \{d \in \Lambda \mid E(d) < \kappa\}, \quad \forall \kappa \ge 0. \tag{2.2}$$

For a closed geodesic c we set $\Lambda(c) = \{\gamma \in \Lambda \mid E(\gamma) < E(c)\}$.

For $m \in \mathbf{N}$ we denote the m-fold iteration map $\phi_m : \Lambda \to \Lambda$ by $\phi_m(\gamma)(t) = \gamma(mt)$, for all $\gamma \in \Lambda, t \in S^1$, as well as $\gamma^m = \phi_m(\gamma)$. If $\gamma \in \Lambda$ is not constant then the multiplicity $m(\gamma)$ of γ is the order of the isotropy group $\{s \in S^1 \mid s \cdot \gamma = \gamma\}$. For a closed geodesic c, the mean index $\hat{i}(c)$ is defined as usual by $\hat{i}(c) = \lim_{m\to\infty} i(c^m)/m$. Using singular homology with rational coefficients we consider the following critical $\mathbf{Q}$-module of a closed geodesic $c \in \Lambda$:

$$\overline{C}_*(E, c) = H_*\left((\Lambda(c) \cup S^1 \cdot c)/S^1, \Lambda(c)/S^1\right). \tag{2.3}$$

The following results of Rademacher will be used in our proofs below.

Proposition 2.1. (cf. Satz 6.11 of [Rad2]) *Let c be a prime closed geodesic on a bumpy Finsler manifold (M, F). Then there holds*

$$\overline{C}_q(E, c^m) = \begin{cases} \mathbf{Q}, & \textit{if} \ \ i(c^m) - i(c) \in 2\mathbf{Z} \ \ \textit{and} \ \ q = i(c^m), \\ 0, & \textit{otherwise}. \end{cases}$$

Definition 2.2. (cf. Definition 1.6 of [Rad1]) *For a closed geodesic c, let $\gamma_c \in \{\pm\frac{1}{2}, \pm 1\}$ be the invariant defined by $\gamma_c > 0$ if and only if $i(c)$ is even, and $|\gamma_c| = 1$ if and only if $i(c^2) - i(c)$ is even.*

Proposition 2.3. (cf. Theorem 3.1 of [Rad1]) *Let $c_k, k = 1, 2, \cdots, r$ prime closed geodesics on a bumpy Finsler 3-sphere. Then the average indices $\hat{i}(c_k)$ and the invariants γ_{c_k} satisfy the identity*

$$\sum_{k=1}^{r} \frac{\gamma_{c_k}}{\hat{i}(c_k)} = 1. \tag{2.4}$$

Let (X, Y) be a space pair such that the Betti numbers $b_i = b_i(X, Y) = \dim H_i(X, Y; \mathbf{Q})$ are finite for all $i \in \mathbf{Z}$. As usual the *Poincaré series* of (X, Y) is defined by the formal power series $P(X, Y) = \sum_{i=0}^{\infty} b_i t^i$. We need the following well known results on Betti numbers and the Morse inequality for $\overline{\Lambda} \equiv \overline{\Lambda} S^3$ and $\overline{\Lambda}^0 = \overline{\Lambda}^0 S^3 = \{\text{constant point curves in } S^3\} \cong S^3$.

Proposition 2.4. (cf. Remark 2.5 of [Rad1]) *The Poincaré series is given by*

$$\begin{aligned} P(\overline{\Lambda} S^3, \overline{\Lambda}^0 S^3)(t) &= t^2 \left(\frac{1}{1 - t^2} + \frac{t^2}{1 - t^2} \right) \\ &= t^2(1 + t^2)(1 + t^2 + t^4 + \cdots) = t^2 + 2t^4 + 2t^6 + \cdots, \end{aligned}$$

which yields

$$b_q = b_q(\overline{\Lambda} S^3, \overline{\Lambda}^0 S^3) = \mathrm{rank} H_q(\overline{\Lambda} S^3, \overline{\Lambda}^0 S^3) = \begin{cases} 1, & \textit{if} \quad q = 2, \\ 2, & \textit{if} \quad q = 2k + 2, \quad k \in \mathbf{N}, \\ 0 & \textit{otherwise}. \end{cases} \tag{2.5}$$

Proposition 2.5. (cf. Theorem I.4.3 of [Cha1], Theorem 6.1 of [Rad2]) *Suppose that there exist only finitely many prime closed geodesics* $\{c_j\}_{1\le j\le k}$ *on a Finsler* 3*-sphere* (S^3, F)*. Set*

$$M_q = \sum_{1\le j\le k,\, m\ge 1} \dim \overline{C}_q(E, c_j^m), \quad \forall q \in \mathbf{Z}.$$

Then for every integer $q \ge 0$ *there holds*

$$\begin{aligned} M_q - M_{q-1} + \cdots + (-1)^q M_0 &\ge b_q - b_{q-1} + \cdots + (-1)^q b_0, && (2.6)\\ M_q &\ge b_q. && (2.7) \end{aligned}$$

2.2 Classification of non-degenerate closed geodesics on S^3

We introduce some notations in [Lon2] here. Given any two real matrices of the square block form

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}_{2i\times 2i}, \qquad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}_{2j\times 2j},$$

we define the $\diamond$-sum of M_1 and M_2 to be the $2(i+j)\times 2(i+j)$ matrix $M_1 \diamond M_2$ given by

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix},$$

and $M_1^{\diamond k}$ to be the k-times $\diamond$-sum of M_1. In the following, let

$$\begin{aligned} N(\alpha, B) &= \begin{pmatrix} \cos\alpha & -\sin\alpha & b_1 & b_2 \\ \sin\alpha & \cos\alpha & b_3 & b_4 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \\ R(\theta) &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad H(d) = \begin{pmatrix} d & 0 \\ 0 & 1/d \end{pmatrix}, \end{aligned}$$

where $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $(b_1, b_2, b_3, b_4) \in \mathbf{R}^4$, θ/π, and $\alpha/\pi \in (0, 2) \setminus \mathbf{Q}$, and $d > 0$ or $d < 0$.

In [Lon1] of 2000, Long defined the *homotopy set* $\Omega(M)$ and the *homotopy component* $\Omega^0(M)$ of M in the symplectic group $\mathrm{Sp}(2n)$ by

$$\Omega(M) = \{N \in \mathrm{Sp}(2n) \mid \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U} \equiv \Gamma \text{ and } \nu_\omega(N) = \nu_\omega(M)\ \forall \omega \in \Gamma\},$$

where $\sigma(M)$ denotes the spectrum of M and $\nu_\omega(M) \equiv \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I)$ for all $\omega \in \mathbf{U}$. Then $\Omega^0(M)$ is defined to be the path connected component of $\Omega(M)$ containing M (cf. also Section 1.8 of [Lon2]).

Let c be a closed geodesic on a Finsler sphere (S^3, F). Denote the linearized Poincaré map of c by P_c. Note that the index iteration formulae in [Lon1] (cf. also [Lon2]) work for Morse indices of iterated closed geodesics on Finsler manifolds (cf. [LLo1]). Suppose that all iterations c^m of c are non-degenerate. Then by Theorems 8.1.4 to 8.1.7 and Theorem 8.3.1 of [Lon2], we have the following classification of non-degenerate closed geodesics, i.e., there exists a path $f_c \in C([0,1], \Omega^0(P_c))$ such that $f_c(0) = P_c$ and $f_c(1)$ have the following forms.

NCG-1. $f_c(1) = R(\theta_1) \diamond R(\theta_2)$.

In this case, by Theorems 8.1.7 and 8.3.1 of [Lon2], we have $i(c) = 2p$ for some $p \in \mathbf{N}_0$, and

$$i(c^m) = 2m(p-1) + 2\sum_{i=1}^{2}\left[\frac{m\theta_i}{2\pi}\right] + 2, \quad \nu(c^m) = 0, \qquad \forall\, m \geq 1. \tag{2.8}$$

NCG-2. $f_c(1) = R(\theta) \diamond H(d)$.

In this case, by Theorems 8.1.6, 8.1.7 and 8.3.1 of [Lon2], we have $i(c) = p$ for some $p \in \mathbf{N}_0$, and

$$i(c^m) = m(p-1) + 2\left[\frac{m\theta}{2\pi}\right] + 1, \quad \nu(c^m) = 0, \qquad \forall\, m \geq 1. \tag{2.9}$$

NCG-3. $f_c(1) = H(d_1) \diamond H(d_2)$.

In this case, by Theorems 8.1.6 and 8.3.1 of [Lon2], we have $i(c) = p$ for some $p \in \mathbf{N}_0$, and

$$i(c^m) = mp, \quad \nu(c^m) = 0, \qquad \forall\, m \geq 1. \tag{2.10}$$

NCG-4. $f_c(1) = N(\alpha, B)$.

In this case, by Theorems 8.2.3, 8.2.4 and 8.3.1 of [Lon2], we have $i(c) = 2p$ for some $p \in \mathbf{N}_0$, and

$$i(c^m) = 2mp, \quad \nu(c^m) = 0, \qquad \forall\, m \geq 1. \tag{2.11}$$

3 Proof of main theorems

Firstly, we list below three auxiliary results. By Theorem 1.2 in [DuL1] (cf. also [Rad4]), there exist at least two prime closed geodesics on every bumpy Finsler sphere (S^n, F) for $n \geq 2$. As also noticed in Rademacher's preprint [Rad4], we point out that the *common index jump theorem* due to Long and Zhu in [LoZ1] also works for closed geodesics on Finsler manifolds (cf. Remark 12.2.5 of [Lon2]). In the bumpy case, we have the following consequence.

Theorem 3.1 (cf. Theorem 4.3 of [LoZ1] and Theorem 11.2.1 of [Lon2]) *Let c be a closed geodesic on a compact bumpy Finsler manifold (M, F) with $\hat{i}(c) > 0$. Then there exist infinitely many $k \in \mathbf{N}$ such that*

$$i(c^{2k+1}) - i(c^{2k-1}) = 2i(c). \tag{3.1}$$

Proof. For readers conveniences, we sketch the proof here. By Theorem 4.3 of [LoZ1] (cf. (11.2.4) and (11.2.5) in Theorem 11.2.1 of [Lon2]) we obtain for infinitely many $k \in \mathbf{N}$,

$$i(c^{2k+1}) - i(c^{2k-1}) - 2i(c) = 2S^+_{P_c}(1) + \nu(c^{2k-1}) - \nu(c),$$

where $S^+_{P_c}(1)$ is the positive splitting number of P_c at 1. Because (M, F) is bumpy, all terms in the right hand side of the above identity are zero. ∎

Lemma 3.2. *Let (M, F) a bumpy Finsler manifold with only finitely many prime closed geodesics. If c is a closed geodesic on (M, F) which is not an absolute minimum of the energy functional E in its free homotopy class, the mean index of the closed geodesic c must satisfy $\hat{i}(c) > 0$. This holds always when M is simply connected, specially a sphere.*

Proof. It is well known that the Morse index sequence $i(c^m)$ either tends to $+\infty$ asymptotically linearly or $i(c^m) = 0$ for all $m \geq 1$. Therefore $\hat{i}(c) = 0$ if and only if $i(c^m) = 0$ for all $m \geq 1$. A crucial point in the proof of this fact is the Property (2) in Proposition 1.3 in [Bot1] of Bott (cf. also Lemma 1 of [GrM1]).

Now let c be a prime closed geodesic on (M, F) with $\hat{i}(c) = 0$ which is not an absolute minimum of E. Because (M, F) is bumpy, every iteration c^m of c is homologically visible, by Theorem 3 of [BaK1] there must exist infinitely many prime closed geodesics on (M, F). ∎

Lemma 3.3 *Let (M, F) a Finsler manifold with only finitely many prime closed geodesics. If the Morse type numbers $M_{2k-1} = 0$ for all $k \in \mathbf{N}$, then $M_q = b_q$ holds for all $q \in \mathbf{N}_0$. Specially, on a bumpy Finsler (S^3, F) with finitely many prime closed geodesics, if the Morse indices of iterations of these prime closed geodesics are all even, $M_q = b_q$ holds for all $q \in \mathbf{N}_0$.*

Proof. It follows directly from Proposition 2.1 and the Morse inequality. ∎

In this section we denote the contribution of iterations of prime closed geodesics c_i to the Morse type number M_q by $M_q(i)$ for $i = 1, 2$ and $q \geq 0$ below. The following lemma is crucial in the proof of our main theorems.

Lemma 3.4. *Let $S^3 = (S^3, F)$ be a bumpy Finsler sphere with precisely two prime closed geodesics c_1 and c_2. Suppose that c_1 and c_2 do not belong to the classes $\{NCG\text{-}1, NCG\text{-}2\}$ simultaneously. Then at least one of the two closed geodesics must satisfy $i(c) = 2$.*

Proof. By the Morse inequality and Proposition 2.4, we have $M_2 \geq b_2 = 1$. Thus at least one of the two closed geodesics must have Morse index $i(c) \leq 2$. Without loss of generality, let $i(c_1) \leq 2$. Assume the Lemma 3.4 does not hold. Then we have $0 \leq i(c_1) \leq 1$ and $i(c_2) \neq 2$. If c_i for $i = 1$ or 2 belongs to the class NCG-3 or NCG-4, then $i(c_i) > 0$ by Lemma 3.2. So by (2.11),

c_1 does not belong to NCG-4. Next we carry out our proof in four cases according to the value of $i(c_1)$ and the classification of c_1.

Case I: *$c_1 \in$NCG-3 with $i(c_1) = 1$.*

By Proposition 2.1, there holds $M_{2k+1}(1) = 1$, $M_{2k}(1) = 0, k \in \mathbf{N}_0$. We continue our study in 5 subcases (i)-(v):

(i) If $c_2 \in$ {NCG-3, NCG-4} with $i(c_2) = 2p \geq 4$. Then by Definition 2.2, we have $\gamma_{c_1} = -\frac{1}{2}$ and $\gamma_{c_2} = 1$. Hence by Proposition 2.3 we obtain $0 > \frac{1}{2p} - \frac{1}{2} = \frac{\gamma_{c_2}}{\hat{i}(c_2)} - \frac{\gamma_{c_1}}{\hat{i}(c_1)} = 1$ contradiction!

(ii) If $c_2 \in$NCG-3 with $i(c_2)$ odd, then by Proposition 2.1, Morse type numbers $M_{2k}(2) = 0, k \in \mathbf{N}_0$. So $M_2 = 0$, which contradicts to $M_2 \geq b_2 = 1$ by Propositions 2.4 and 2.5.

(iii) If $c_2 \in$NCG-2 with $i(c_2) = p \geq 0$, then $\hat{i}(c_2) = p - 1 + \frac{\theta}{\pi}$ is an irrational number by the definition of θ. Hence $\sum_{i=1}^{2} \frac{\gamma_{c_i}}{\hat{i}(c_i)}$ is also an irrational number. However, by Proposition 2.3 we have $\sum_{i=1}^{2} \frac{\gamma_{c_i}}{\hat{i}(c_i)} = 1$ contradiction!

(iv) If $c_2 \in$NCG-1 with $i(c_2) = 2p \geq 4$, then by Proposition 2.1, $M_2(2) = 0$. So $M_2 = M_2(1) + M_2(2) = 0$, which contradicts to $M_2 \geq b_2 = 1$ by Propositions 2.4 and 2.5.

(v) If $c_2 \in$ NCG-1 with $i(c_2) = 0$, then we have $\hat{i}(c_2) = \frac{\theta_1}{\pi} + \frac{\theta_2}{\pi} - 2$, $\hat{i}(c_1) = 1$ by (2.8) and (2.10) in Section 2. By Definition 2.2 we obtain $\gamma_{c_1} = -\frac{1}{2}$ and $\gamma_{c_2} = 1$. Then by Proposition 2.3, we obtain the identity

$$\frac{\theta_1}{\pi} + \frac{\theta_2}{\pi} = \frac{8}{3}. \tag{3.2}$$

On the other hand, in this subcase we have $M_{2k-1}(2) = 0$ for $k \in \mathbf{N}$. Thus

$$M_{2k-1} = M_{2k-1}(1) + M_{2k-1}(2) = 1, \quad \forall k \in \mathbf{N}. \tag{3.3}$$

Claim 1: *$i(c_2) = 0$, $i(c_2^2) = i(c_2^3) = 2$, $i(c_2^4) = i(c_2^5) = i(c_2^6) = 4$ and $i(c_2^7) = 6$.*

In fact, by (3.2) we have $\sum_{i=1}^{2} \frac{\theta_i}{\pi} < 3$. Noting that $i(c_2^m)$ is even, it follows from $i(c_2^2) = 2\sum_{i=1}^{2} \left[\frac{\theta_i}{\pi}\right] - 2 \leq 2$ that $i(c_2^2) = 0$ or 2. If $i(c_2^2) = 0$, then $M_0 = M_0(2) \geq 2$. However it follows from Propositions 2.4, 2.5 and (3.3) that $-1 \geq M_1 - M_0 \geq b_1 - b_0 = 0$, which is a contradiction. So $i(c_2^2) = 2$. By (3.2), we have $\frac{3\theta_1}{2\pi} + \frac{3\theta_2}{2\pi} = 4$. Since both $\frac{3\theta_1}{2\pi}$ and $\frac{3\theta_2}{2\pi}$ are irrational numbers by the definition of $\theta_i, i = 1, 2$, we have $\sum_{i=1}^{2} [\frac{3\theta_i}{2\pi}] \leq 3$. And so $i(c_2^3) = 2\sum_{i=1}^{2} \left[\frac{3\theta_i}{2\pi}\right] - 4 \leq 2$. By the similar argument it yields $i(c_2^3) = 2$.

By (3.2), we have $\sum_{i=1}^{2} \frac{k\theta_i}{2\pi} = \frac{4k}{3} = k + 1 + \frac{k-3}{3}$ for $k = 4, 5, 6$. Since both $\frac{k\theta_1}{2\pi}$ and $\frac{k\theta_2}{2\pi}$ are irrational numbers, we have $\sum_{i=1}^{2} [\frac{k\theta_i}{2\pi}] \leq k + 1$. And so $i(c_2^k) = 2\sum_{i=1}^{2} \left[\frac{k\theta_i}{2\pi}\right] - 2k + 2 \in \{0, 2, 4\}$. If $i(c_2^k) = 0$ or 2 for some $k \in \{4, 5, 6\}$, then $M_0 \geq 2$ or $M_2 \geq 3$. By Propositions 2.4, 2.5 and (3.3) we have $-2 \geq M_3 - M_2 + M_1 - M_0 \geq b_3 - b_2 + b_1 - b_0 = -1$, which is a contradiction. So $i(c_2^4) = i(c_2^5) = i(c_2^6) = 4$.

By (3.2), we have $\frac{7\theta_1}{2\pi}+\frac{7\theta_2}{2\pi}=\frac{28}{3}$. Since both $\frac{7\theta_1}{2\pi}$ and $\frac{7\theta_2}{2\pi}$ are irrational numbers, we have $\sum_{i=1}^2[\frac{7\theta_i}{2\pi}]\le 9$. And so $i(c_2^7)=2\sum_{i=1}^2\left[\frac{7\theta_i}{2\pi}\right]-12\le 6$. By the above argument we then get $i(c_2^7)=6$. Claim 1 is proved.

Next we will estimate values of $\{\frac{\theta_1}{\pi},\frac{\theta_2}{\pi}\}$ by analyzing $i(c_2^m)$ for $m=2,4,5,6,7$ respectively.

By Claim 1, we obtain $2=i(c_2^2)=2\sum_{i=1}^2\left[\frac{\theta_i}{\pi}\right]-2$ which implies

$$\sum_{i=1}^2\left[\frac{\theta_i}{\pi}\right]=2. \tag{3.4}$$

By (3.2), without loss of generality, we get $\frac{\theta_1}{\pi}<\frac{4}{3}<\frac{\theta_2}{\pi}$. By the definition of θ_2, we have $\frac{\theta_2}{\pi}<2$. Then by (3.4), $\frac{\theta_1}{\pi}>1$. In summary, we obtain

$$1<\frac{\theta_1}{\pi}<\frac{4}{3}\quad\text{and}\quad\frac{4}{3}<\frac{\theta_2}{\pi}<2. \tag{3.5}$$

By Claim 1 we have $4=i(c_2^4)=2\sum_{i=1}^2\left[2\frac{\theta_i}{\pi}\right]-6$, which implies

$$\sum_{i=1}^2\left[2\frac{\theta_i}{\pi}\right]=5. \tag{3.6}$$

Hence by (3.5) we have

$$2<2\frac{\theta_1}{\pi}<\frac{8}{3}\quad\text{and}\quad\frac{8}{3}<2\frac{\theta_2}{\pi}<4. \tag{3.7}$$

By (3.7) we have $\left[2\frac{\theta_1}{\pi}\right]=2$. So $\left[2\frac{\theta_2}{\pi}\right]=3$ by (3.6), which, together with (3.7), implies $\frac{3}{2}<\frac{\theta_2}{\pi}<2$. And hence $1<\frac{\theta_1}{\pi}<\frac{7}{6}$ by (3.2). In summary, by the value of $i(c_2^4)$ we obtain the estimates

$$1<\frac{\theta_1}{\pi}<\frac{7}{6}\quad\text{and}\quad\frac{3}{2}<\frac{\theta_2}{\pi}<2. \tag{3.8}$$

By Claim 1 we have $4=i(c_2^5)=2\sum_{i=1}^2\left[\frac{5\theta_i}{2\pi}\right]-8$, which implies

$$\sum_{i=1}^2\left[\frac{5\theta_i}{2\pi}\right]=6. \tag{3.9}$$

Multiplying (3.8) by $\frac{5}{2}$ yields

$$\frac{5}{2}<\frac{5\theta_1}{2\pi}<\frac{35}{12}\quad\text{and}\quad\frac{15}{4}<\frac{5\theta_2}{2\pi}<5. \tag{3.10}$$

By (3.10) we have $\left[\frac{5\theta_1}{2\pi}\right]=2$. So $\left[\frac{5\theta_2}{2\pi}\right]=4$ by (3.9), which, together with (3.10), implies $\frac{8}{5}<\frac{\theta_2}{\pi}<2$. And hence $1<\frac{\theta_1}{\pi}<\frac{16}{15}$ by (3.2). In summary, by the value of $i(c_2^5)$ we obtain the estimates

$$1<\frac{\theta_1}{\pi}<\frac{16}{15}\quad\text{and}\quad\frac{8}{5}<\frac{\theta_2}{\pi}<2. \tag{3.11}$$

By Claim 1 we have $4 = i(c_2^6) = 2\sum_{i=1}^{2}\left[3\frac{\theta_i}{\pi}\right] - 10$, which implies

$$\sum_{i=1}^{2}\left[3\frac{\theta_i}{\pi}\right] = 7. \tag{3.12}$$

Hence by (3.11) we have

$$3 < 3\frac{\theta_1}{\pi} < \frac{16}{5} \quad \text{and} \quad \frac{24}{5} < 3\frac{\theta_2}{\pi} < 6. \tag{3.13}$$

By (3.13) we have $\left[3\frac{\theta_1}{\pi}\right] = 3$. So $\left[3\frac{\theta_2}{\pi}\right] = 4$ by (3.12), which, together with (3.13), implies $\frac{8}{5} < \frac{\theta_2}{\pi} < \frac{5}{3}$. In summary, by the value of $i(c_2^6)$ we obtain estimates

$$1 < \frac{\theta_1}{\pi} < \frac{16}{15} \quad \text{and} \quad \frac{8}{5} < \frac{\theta_2}{\pi} < \frac{5}{3}. \tag{3.14}$$

By Claim 1 we have $6 = i(c_2^7) = 2\sum_{i=1}^{2}\left[\frac{7\theta_i}{2\pi}\right] - 12$, which implies

$$\sum_{i=1}^{2}\left[\frac{7\theta_i}{2\pi}\right] = 9. \tag{3.15}$$

Hence by (3.14) we have

$$\frac{7}{2} < \frac{7\theta_1}{2\pi} < \frac{56}{15} < 4 \quad \text{and} \quad \frac{28}{5} < \frac{7\theta_2}{2\pi} < \frac{35}{6}. \tag{3.16}$$

By (3.16) we have $\left[\frac{7\theta_1}{2\pi}\right] = 3$. So $\left[\frac{7\theta_2}{2\pi}\right] = 6$ by (3.15), which, together with (3.16), implies $6 < \frac{7\theta_2}{2\pi} < \frac{35}{6}$. This leads to a contradiction.

Case II: $c_1 \in$*NCG-2 with* $i(c_1) = 0$ *or* 1.

Then in this case $\hat{i}(c_1)$ is an irrational number by the definition of θ. By the assumption, c_1 and c_2 do not simultaneously belong to the classes {NCG-1, NCG-2}. Then $c_2 \in$ {NCG-3, NCG-4} with $\hat{i}(c_2)$ an integer, Hence $\sum_{i=1}^{2}\frac{\gamma_{c_i}}{\hat{i}(c_i)}$ is an irrational number. However, by Proposition 2.3 we have $\sum_{i=1}^{2}\frac{\gamma_{c_i}}{\hat{i}(c_i)} = 1$ contradiction!

Case III: $c_1 \in$*NCG-1 with* $i(c_1) = 0$.

We continue our study 3 subcases:

(i) If $c_2 \in$ {NCG-3, NCG-4} with $i(c_2)$ even, then the condition of Lemma 3.3 is satisfied, i.e., $M_0 = b_0 = 0$ by Proposition 2.4. However, in this subcase we have $M_0 \geq 1$, contradiction!

(ii) If $c_2 \in$NCG-3 with $i(c_2) = 2p - 1 \geq 3$, then in this subcase we have $M_0 \geq 1$ and $M_1 = 0$. But by Propositions 2.4 and 2.5 we obtain $-1 \geq M_1 - M_0 \geq b_1 - b_0 = 0$, contradiction!

(iii) If $c_2 \in$NCG-3 with $i(c_2) = 1$, then this is exactly subcase (v) in Case I.

This completes the proof of Lemma 3.4. ∎

Remark 3.5. From the proof of Lemma 3.4, one can see that, when there exist precisely two prime closed geodesics c_1 and c_2 on a bumpy (S^3, F), at least one of them must have initial index 2 provided they do not belong to the following two precise classes:

(1) $c_1 \in$NCG-1 with $i(c_1) = 0$ and $c_2 \in$NCG-2 with $i(c_2) = 1$.

(2) $c_1 \in$NCG-2 with $i(c_1) = 0$ and $c_2 \in$NCG-2 with $i(c_2) = 1$.

Note that Theorem 1.1 is a weaker consequence of the following Theorem 3.6.

Theorem 3.6. *Let (M, F) be a bumpy Finsler $\mathbf{Q}$-homological S^3 with precisely two prime closed geodesics c_1 and c_2. Then both of c_1 and c_2 must belong to classes $\{NCG\text{-}1, NCG\text{-}2\}$.*

Proof of Theorem 3.6. Because the proof is the same for $\mathbf{Q}$-homological S^3, it suffices to prove Theorem 3.6 for bumpy Finsler $S^3 = (S^3, F)$ only. Suppose that there exist precisely two closed geodesics c_1 and c_2 on (S^3, F). Assume the theorem does not hold, without loss of generality, we can suppose $i(c_1) = 2$ by Lemma 3.4. Next we carry out our proof in three steps according to the classification of c_1, and will derive some contradiction in each case.

Step 1: $c_1 \in \{NCG\text{-}3,\ NCG\text{-}4\}$

In this case $i(c_1^m) = 2m, \forall m \geq 1$. Then by Proposition 2.1, there holds $M_{2k}(1) = 1$, $M_{2k-1}(1) = 0, \forall k \in \mathbf{N}$. Since $M_{2k} = M_{2k}(1) + M_{2k}(2) \geq 2, \forall k \geq 2$ by Propositions 2.4 and 2.5, it yields $M_{2k}(2) \geq 1, \forall k \geq 2$. So $i(c_2)$ must be even by Proposition 2.1. We continue our study in 4 subcases:

(i) If $c_2 \in \{\text{NCG-3, NCG-4}\}$ with $i(c_2^m) = 2m, m \in \mathbf{N}$, then $M_2(2) = 1$. Thus $2 = M_2 = b_2 = 1$ by Proposition 2.4 and Lemma 3.3, contradiction!

(ii) If $c_2 \in \{\text{NCG-3, NCG-4}\}$ with $i(c_2) = 2p \geq 4$, then we have $\gamma_{c_1} = \gamma_{c_2} = 1$. Hence by Proposition 2.3 we obtain $1 = \sum_{k=1}^{2} \frac{\gamma(c_k)}{\hat{i}(c_k)} = \frac{1}{2} + \frac{1}{2p} < 1$, contradiction!

(iii) If $c_2 \in$ NCG-2, noting that $\hat{i}(c_2)$ is an irrational number and $\hat{i}(c_1)$ is an integer, this leads to a contradiction by Proposition 2.3.

(iv) If $c_2 \in$NCG-1, then we have $\hat{i}(c_2) = 2(p-1) + \frac{\theta_1+\theta_2}{\pi}$. And by Definition 2.2, $\gamma_{c_1} = \gamma_{c_2} = 1$. Hence by Proposition 2.3 we obtain $\frac{1}{2(p-1)+\frac{\theta_1+\theta_2}{\pi}} + \frac{1}{2} = 1$, i.e., $\frac{\theta_1+\theta_2}{\pi} = 4 - 2p$. By the definitions of θ_1 and θ_2, we have $\frac{\theta_1+\theta_2}{\pi} \in (0, 4)$. So $p = 1$, which implies $M_2(2) \geq 1$. Thus $2 \leq M_2(1) + M_2(2) = M_2 = b_2 = 1$, contradiction!

Step 2: $c_1 \in$*NCG-2 with* $i(c_1) = 2$.

In this case $i(c_1^m) = m + 2[\frac{m\theta}{2\pi}] + 1, \forall m \geq 1$. Hence by Proposition 2.1, $M_{2k}(1) = 1$ for some $k \in \mathbf{N}$ and $M_{2j-1}(1) = 0$ for $j \in \mathbf{N}$. We continue our study in 8 subcases:

(i) If $c_2 \in \{\text{NCG-3, NCG-4}\}$, noting that $\hat{i}(c_1)$ is an irrational number and $\hat{i}(c_2)$ is an integer, this leads to a contradiction by Proposition 2.3.

(ii) If $c_2 \in$NCG-2 with $i(c_2)$ odd, then $M_{2k}(2) = 0, k \in \mathbf{N}$. So we have $M_{2k} \leq 1, \forall k \geq 2$, which contradicts to $M_{2k} \geq b_{2k} = 2, \forall k \geq 2$ by Propositions 2.4 and 2.5.

(iii) If $c_2 \in$NCG-2 with $i(c_2) = 0$, then $M_0 \geq 1$ and $M_{2k-1} = 0, \forall k \in \mathbf{N}$. So by Proposition 2.4 and Lemma 3.3 we obtain $1 \leq M_0 = b_0 = 0$, contradiction!

(iv) If $c_2 \in$NCG-2 with $i(c_2) = 2$, then $M_2 = 2$ and $M_{2k-1} = 0, \forall k \in \mathbf{N}$. So by Proposition 2.4 and Lemma 3.3 we obtain $2 = M_2 = b_2 = 1$, contradiction!

(v) If $c_2 \in$NCG-2 with $i(c_2) = 2p \geq 4$, $i(c_2^m) = m(2p-1) + 2[\frac{m\theta}{2\pi}] + 1, \forall m \geq 1$. Let $2T = 2(2p-1) + 2[\frac{\theta}{\pi}] + 2$. Then $2 < 2T \notin \{i(c_2^m) \,|\, m \in \mathbf{N}\}$. So by Proposition 2.1 $M_{2T}(2) = 0$, which implies $M_{2T} = M_{2T}(1) + M_{2T}(2) \leq 1$, where $2T > 2$. By Lemma 3.3 it yields $1 \geq M_{2T} = b_{2T} = 2$, contradiction!

(vi) If $c_2 \in$NCG-1 with $i(c_2) = 0, i(c_2^m) = -2m + 2\sum_{i=1}^{2} \left[\frac{m\theta_i}{2\pi}\right] + 2, m \in \mathbf{N}$, then by Proposition 2.1 and Lemma 3.3, we have $1 \leq M_0 = b_0 = 0$, contradiction!

(vii) If $c_2 \in$NCG-1 with $i(c_2) = 2$, then $M_2 = 2$ and $M_{2k-1} = 0, \forall k \in \mathbf{N}$. So by Proposition 2.4 and Lemma 3.3 we obtain $2 = M_2 = b_2 = 1$, contradiction!

(viii) If $c_2 \in$NCG-1 with $i(c_2) = 2k \geq 4$, then $i(c_2^m) = 2m(k-1) + 2\sum_{i=1}^{2} \left[\frac{m\theta_i}{2\pi}\right] + 2, \forall m \geq 1$ and $M_{2j-1}(2) = 0, \forall j \in \mathbf{N}$. Let $m_0 = \min\{m \in \mathbf{N} | \sum_{i=1}^{2} \left[\frac{m\theta_i}{2\pi}\right] \geq 1\}$. Because $\sum_{i=1}^{2}[\frac{\theta_i}{2\pi}] = 0$ by the definitions of $\frac{\theta_i}{\pi}$, $m_0 \geq 2$. Then

$$
\begin{aligned}
i(c_1^{m_0-1}) &= 2(m_0-1)(k-1) + 2. && (3.17)\\
i(c_1^{m_0}) &= 2m_0(k-1) + 2 + 2\sum_{i=1}^{2} \left[\frac{m_0\theta_i}{2\pi}\right] \\
&\geq i(c_1^{m_0-1}) + 4. && (3.18)
\end{aligned}
$$

Let $2T = i(c_1^{m_0-1}) + 2$. Notice that $i(c_1^m)$ is non-decreasing, $2T \notin \{i(c_2^m), m \in \mathbf{N}\}$. So by Proposition 2.1 we have $M_{2T}(2) = 0$, where $2T \geq 4$. So we have $M_{2T} \leq 1$. By Lemma 3.3 we have $1 \geq M_{2T} = b_{2T} = 2$, contradiction!

Step 3: $c_1 \in$*NCG-1 with* $i(c_1) = 2$.

In this case, we have $\hat{i}(c_1) = \frac{\theta_1}{\pi} + \frac{\theta_2}{\pi}$ and

$$
i(c_1^m) = 2\sum_{i=1}^{2} \left[\frac{m\theta_i}{2\pi}\right] + 2, \forall m \geq 1 \qquad \text{and} \qquad M_{2k-1}(1) = 0, \forall k \in \mathbf{N}. \tag{3.19}
$$

We continue our study in 4 subcases:

(i) If $c_2 \in \{$NCG-3,NCG-4$\}$ with $i(c_2^m) = 2m, m \in \mathbf{N}$, then $M_2 = M_2(1) + M_2(2) = 2$, which is a contradiction to Proposition 2.4 and Lemma 3.3.

(ii) If $c_2 \in \{\text{NCG-3,NCG-4}\}$ with $i(c_2^m) = 2pm \geq 4, m \in \mathbf{N}$ and $\hat{i}(c_2) = 2p$, then we have

$$M_{2p}(2) = 1, \quad M_q(2) = 0,\ 0 \leq q \leq 2p-1, \qquad M_{2k-1}(2) = 0, \quad \forall k \in \mathbf{N}. \tag{3.20}$$

By Definition 2.2 we have $\gamma_{c_1} = \gamma_{c_2} = 1$. So it follows from Proposition 2.3 in Section 2 that $\frac{1}{\frac{\theta_1}{\pi}+\frac{\theta_2}{\pi}} + \frac{1}{2p} = 1$, i.e., we have

$$\frac{2p-1}{2}(\frac{\theta_1}{\pi} + \frac{\theta_2}{\pi}) = p. \tag{3.21}$$

Noting that both $\frac{(2p-1)\theta_1}{2\pi}$ and $\frac{(2p-1)\theta_2}{2\pi}$ are irrational numbers, we obtain $[\frac{(2p-1)\theta_1}{2\pi}] + [\frac{(2p-1)\theta_2}{2\pi}] \in \{0, 1, \cdots, p-1\}$. Hence we have

$$i(c_1^{2p-1}) = 2\left(\left[\frac{(2p-1)\theta_1}{2\pi}\right] + \left[\frac{(2p-1)\theta_2}{2\pi}\right]\right) + 2 \in \{2, 4, \cdots, 2p\}. \tag{3.22}$$

Claim 2: $i(c_1^{2p-2}) = i(c_1^{2p-1}) = 2p$.

In fact, by Proposition 2.4, Lemma 3.3 and (3.20), we have

$$M_2(1) = 1, \text{ and if } 2p-2 \geq 4, \text{ then } M_{2q}(1) = 2,\ 4 \leq 2q \leq 2p-2. \tag{3.23}$$

If $2p = 4$, then by (3.19) and (3.21) it yields $i(c_1^2) \leq 4$. We claim $i(c_1^2) = 4$. Otherwise, assume $i(c_1^2) = 2$, then it yields $M_2 = M_2(1) = 2$, contradicting (3.23). So we have $i(c_1^2) = 4$. Since $i(c_1^3) \geq i(c_1^2) = 4$, by (3.22) we obtain $i(c_1^3) = 4$.

If $2p-2 \geq 4$, noting that $i(c_1^m)$ is non-decreasing, then by (3.23), M_2 is uniquely contributed by c_1, and $M_{2q}(1)$ in (3.23) should be uniquely contributed by the two successive iterations $c_1^{m_0}$ and $c_1^{m_0+1}$ with $i(c_1^{m_0}) = i(c_1^{m_0+1}) = 2q$ for some $m_0 \in \mathbf{N}$. So we have the following sequence about the values of $i(c_1^m), 1 \leq m \leq 2p-2$

$$i(c_1) = 2, \tag{3.24}$$
$$i(c_1^{2j}) = i(c_1^{2j+1}) = 2j+2, \text{ for } j = 1, \cdots, p-2, \tag{3.25}$$
$$i(c_1^{2p-2}) = 2p. \tag{3.26}$$

Since $i(c_1^{2p-1}) \geq i(c_1^{2p-2})$, by (3.22) and (3.26) we obtain $i(c_1^{2p-1}) = 2p$. Claim 2 is proved.

By Claim 2, it yields $M_{2p}(1) \geq 2$. So $M_{2p} = M_{2p}(1) + M_{2p}(2) \geq 3$, where $2p \geq 4$. However, by Proposition 2.4 and Lemma 3.3 we have $3 = M_{2p} = b_{2p} = 2$, contradiction!

(iii) If $c_2 \in \{\text{NCG-2, NCG-3}\}$ with $i(c_2)$ odd, then $M_{2k}(2) = 0, k \in \mathbf{N}$ by Proposition 2.1. But by Propositions 2.4 and 2.5, we have $M_2 \geq 1$ and $M_{2k} \geq 2, \forall k \geq 2$. Hence Morse type numbers M_{2k} must be contributed by iterations of c_1, i.e., $M_{2k} = M_{2k}(1), k \in \mathbf{N}$. Thus M_2 should be contributed at least by c_1, and $M_{2k}, k \geq 2$ should be contributed at least by the two successive iterations $c_1^{m_0}$

and $c_1^{m_0+1}$ with $i(c_1^{m_0}) = i(c_1^{m_0+1}) = 2k$ for some $m_0 \in \mathbf{N}$. Because $i(c_1^m)$ is non-decreasing, we have

$$i(c_1^{k+2}) - i(c_1^k) \in \{0, 2\}, \ \forall k \in \mathbf{N}. \tag{3.27}$$

But by the common index jump Theorem 3.1, there exist infinitely many $k' \in \mathbf{N}$ such that

$$i(c_1^{2k'+1}) - i(c_1^{2k'-1}) = 2i(c_1) = 4, \tag{3.28}$$

contradicting (3.27).

(iv) If $i(c_2) \in$NCG-2 with $i(c_2) = 2p \geq 4, i(c^m) = m(2p-1) + 2\left[\frac{m\theta}{2\pi}\right] + 1, m \in \mathbf{N}$ and $\hat{i}(c_2) = 2p - 1 + \frac{\theta}{\pi}$. By Definition 2.2 $\gamma_{c_1} = \gamma_{c_2} = 1$. So by Proposition 2.3, we have $\frac{1}{\frac{\theta_1}{\pi}+\frac{\theta_2}{\pi}} + \frac{1}{2p-1+\frac{\theta}{\pi}} = 1$, i.e.,

$$\frac{\theta_1}{\pi} + \frac{\theta_2}{\pi} < \frac{2p-1}{2p-2} \leq \frac{3}{2}. \tag{3.29}$$

By (3.29) we obtain $[\frac{\theta_1}{\pi}] + [\frac{\theta_2}{\pi}] \in \{0, 1\}$. If $[\frac{\theta_1}{\pi}] + [\frac{\theta_2}{\pi}] = 0$, then $i(c_1^2) = 2([\frac{\theta_1}{\pi}] + [\frac{\theta_2}{\pi}]) + 2 = 2$. So $M_2 = M_2(1) = 2$. However, by Proposition 2.4 and Lemma 3.3 we have $2 = M_2 = b_2 = 1$, contradiction! Hence $[\frac{\theta_1}{\pi}] + [\frac{\theta_2}{\pi}] = 1$, which together with (3.29) yields, without loss of generality,

$$1 < \frac{\theta_1}{\pi} < \frac{2p-1}{2p-2}. \tag{3.30}$$

Claim 3: *$i(c_1^{2p-1}) = 2p + 2$ and $i(c_1^{4p-2}) = 4p$.*

In fact, in this subcase we have $M_{2k-1} = 0, k \in \mathbf{N}$ by Proposition 2.1. So the condition of Lemma 3.3 is satisfied. Noting that $i(c_2) = 2p, i(c_2^2) \in 2\mathbf{Z} - 1$ and $i(c_2^m) \geq 3(2p-1) + 1 > 4p, \forall m \geq 3$, by Proposition 2.1 we obtain

$$M_{2p}(2) = 1, \ M_q(2) = 0, \ \forall\ 0 \leq q \leq 4p, \ q \neq 2p. \tag{3.31}$$

Hence Proposition 2.4, Lemma 3.3 and (3.31) yield

$$M_2(1) = M_{2p}(1) = 1, \ M_{2q}(1) = 2, \ \ \forall\ 4 \leq 2q \leq 4p, \ 2q \neq 2p. \tag{3.32}$$

Noting that $i(c_1^m)$ is non-decreasing, so by (3.32) we have the following

$$\begin{aligned} &\{i(c_1), i(c_1^2), i(c_1^3) \cdots, i(c_1^{x-3}), i(c_1^{x-2}), i(c_1^{x-1}), i(c_1^x), i(c_1^{x+1}) \cdots i(c_1^{y-1}), i(c_1^y)\} \\ &= \{2, 4, 4, \cdots, 2p-2, 2p-2, 2p, 2p+2, 2p+2, \cdots, 4p, 4p\}, \end{aligned} \tag{3.33}$$

where $x = \min\{m \in \mathbf{N} \,|\, i(c_1^m) = 2p+2\}$ and $y = \max\{m \in \mathbf{N} \,|\, i(c_1^m) = 4p\}$, which are determined by the equations

$$x = 2 \cdot \frac{2p-4}{2} + 3 = 2p - 1. \tag{3.34}$$

$$y = 2 \cdot \frac{2p-4}{2} + 1 + 2 \cdot \frac{4p-2p}{2} + 1 = 4p - 2. \tag{3.35}$$

So by (3.33), (3.34) and (3.35), Claim 3 is proved.

By (3.30), we obtain $\frac{4p-2}{2}\frac{\theta_1}{\pi} = (2p-1)\frac{\theta_1}{\pi} > 2p-1$. So by Claim 3 and (3.19), it yields

$$4p = i(c_1^{4p-2}) = 2\sum_{i=1}^{2}\left[\frac{(4p-2)\theta_i}{2\pi}\right] + 2 \geq 4p. \tag{3.36}$$

Thus $\frac{4p-2}{2}\frac{\theta_1}{\pi} < 2p$, i.e.,

$$\frac{\theta_1}{\pi} < \frac{2p}{2p-1}. \tag{3.37}$$

On the other hand, by (3.29) and (3.30), it yields $\frac{\theta_2}{\pi} < \frac{2p-1}{2p-2} - 1 = \frac{1}{2p-2}$. Since $2p \geq 4$, we have

$$\frac{(2p-1)\theta_2}{2\pi} < \frac{2p-1}{2(2p-2)} < 1. \tag{3.38}$$

Thus by Claim 3 and (3.19), it yields

$$2p+2 = i(c_1^{2p-1}) = 2\sum_{i=1}^{2}\left[\frac{(2p-1)\theta_i}{2\pi}\right] + 2 = 2\left[\frac{(2p-1)\theta_1}{2\pi}\right] + 2, \tag{3.39}$$

which implies $p < \frac{2p-1}{2}\frac{\theta_1}{\pi} < p+1$. This is to say

$$\frac{\theta_1}{\pi} > \frac{2p}{2p-1}. \tag{3.40}$$

contradicting (3.37).

Above three steps complete the proof of Theorem 3.6. ∎

Remark 3.7. Suppose that there exist precisely two prime closed geodesics c_1 and c_2 on a bumpy (S^3, F). From Remark 3.5 and the proof of Theorem 3.6, both c_1 and c_2 are non-hyperbolic and must belong to one of the following precise classes:

(I) $c_1 \in$NCG-1 with $i(c_1) = 0$ and $c_2 \in$NCG-2 with $i(c_2) = 1$.

(II) $c_1 \in$NCG-2 with $i(c_1) = 0$ and $c_2 \in$NCG-2 with $i(c_2) = 1$.

(III) $c_1 \in$NCG-1 with $i(c_1) = 2$ and $c_2 \in$NCG-1 with $i(c_2) = 2p \geq 4$.

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